**Adaptive Delayed Feedback Control of Uncertain Fractional Order Chaotic Systems Using Sliding Mode Control**

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**Abstract**

This paper presents an adaptive delayed feedback control method for stabilizing unstable periodic orbits (UPOs) of uncertain fractional order chaotic systems using sliding mode control. The main goal of this research is to develop an adaptive control method based on Lyapunov approach and sliding mode techniques such that the closed loop control system be asymptotically stable on a periodic trajectory which can be sufficiently close to the UPO of the fractional chaotic system. Robustness of the closed loop control system against the system uncertainty and external disturbances is guaranteed. Finally, the proposed method is used to stabilize the UPO of fractional order Duffing system and the numerical simulations show the effectiveness of this method.

**Keywords**: Fractional order systems; Chaos, Nonlinear delayed feedback, Sliding mode Control.

1. **Introduction**

Chaos is a phenomenon that appears in nonlinear dynamical systems and has been observed in many fields of science such as economics [[1](#_ENREF_1)], chemistry [[2](#_ENREF_2)], biology [[3](#_ENREF_3)], engineering [[4](#_ENREF_4)], and so on. In recent years, control of chaotic systems have attracted many scientists in variety of fields. One of the best known methods for chaos control is stabilizing an unstable periodic orbit (UPO) [[5](#_ENREF_5), [6](#_ENREF_6)]. The first method for stabilizing the UPO of chaotic systems was introduced in 1990 is named OGY [[7](#_ENREF_7)]. After that time several extensions of the above-mentioned method were introduced [[8](#_ENREF_8), [9](#_ENREF_9)]. Linear delayed feedback method proposed for control of chaotic systems by Pyragas. The best-known advantage of this method is that it does not need any information about the periodic solutions other than its period [[10](#_ENREF_10)]. The delayed feedback control scheme is used for control of both discrete time systems [[11](#_ENREF_11), [12](#_ENREF_12)] and continuous time systems [[13](#_ENREF_13), [14](#_ENREF_14)]. Up to this time, several methods have been presented for the control of chaotic systems such as: feedback linearization [[15](#_ENREF_15), [16](#_ENREF_16)], sliding mode control [[17](#_ENREF_17), [18](#_ENREF_18)], fuzzy control [[19](#_ENREF_19), [20](#_ENREF_20)], variable structure control [[21](#_ENREF_21)], backstepping [[22](#_ENREF_22)], minimum entropy control [[23](#_ENREF_23)], and fuzzy minimum entropy control [[24](#_ENREF_24)].

Fractional calculus is an old field of mathematics that has had no real applications over many years, but recently, it has been extensively used in variety of fields of science and engineering. It has been used to model many systems in several fields e.g. electromagnetics problems [[25](#_ENREF_25), [26](#_ENREF_26)], viscoelastic damping [[27](#_ENREF_27)], signal processing [[28](#_ENREF_28)], vibration [[29](#_ENREF_29)], diffusion wave [[30](#_ENREF_30)], control theory [[31](#_ENREF_31), [32](#_ENREF_32)] and chaos control [[33-35](#_ENREF_33)]. In addition, by developing fractional calculus in control theory, many fractional order controllers are introduced like, fractional PID controller [[36](#_ENREF_36)], fractional PI controller [[37](#_ENREF_37)], fractional PD controller [[38](#_ENREF_38)], fractional lead-lag controller [[39](#_ENREF_39), [40](#_ENREF_40)], fractional CRONE controller [[41](#_ENREF_41)] and so on.

In recent years, different techniques have been developed to achieve the fractional order chaotic systems control. For example, Lyapunov-based control [[42](#_ENREF_42)], linear feedback control [[43](#_ENREF_43)], sliding mode control [[17](#_ENREF_17), [44](#_ENREF_44)], fuzzy control [[45](#_ENREF_45)], and intelligent control [[46](#_ENREF_46)]. In most of the mentioned methods, the exact model of the systems is needed to derive controllers, but in many applications, because of the system uncertainty and external disturbances, mathematical model of the system is not known.

After development of fractional calculus in dynamical systems, it has been shown that chaotic systems in fractional order form can behave like integer order ones. Afterwards, it is demonstrated that UPO can be found in these systems as it can be shown in integer order ones [[47](#_ENREF_47), [48](#_ENREF_48)]. This paper presents an adaptive nonlinear delayed feedback control via sliding mode control for stabilizing a periodic orbit with known period, which is near to the UPO of the fractional chaotic system.

The main contribution of the paper can be itemized as:

1. The proposed method can stabilize the UPO of a fractional order chaotic system even when the trajectory of UPO is not known. In this case the period of the UPO should be known.
2. The Pyragas method is one the most well-known approaches which can be applied to stabilize the UPO when only the period of UPO is known. This method is also called delayed feedback control; however, this method has not a systematic procedure especially when a continuous-time chaotic system is aimed to be controlled, so its feedback gains should be determined by trial and error. In this paper a systematic algorithm for delayed feedback control of fractional order continuous-time chaotic systems is proposed. The proposed method is nonlinear and, it is applicable to many nonlinear fractional chaotic systems.
3. The proposed method is a fractional order version of [[49](#_ENREF_49)]. In this work the uncertainties and disturbances are assumed to have unknown bounds, and so an adaptive method is suggested.

This paper is organized as follows: Section 2 is a review of the preliminary concepts of fractional calculus. In Section 3 the problem statement is presented. In Section 4, an adaptive sliding mode control method is introduced. The parameters are updated using an adaptation mechanism. In Section 5, the proposed scheme is utilized for stabilizing chaotic Duffing system. Simulation results confirm the effectiveness of the presented method. Finally, a brief conclusion is presented in the latest section.

1. **Preliminary Concepts**

There are several definitions of fractional derivatives. The best known definitions are Riemann-Liouville, Grünwald-Letnikov and Caputo.

**Definition 1.** (Riemann-Liouville Fractional Integral [[50](#_ENREF_50)])**.** The  order fractional integral of function  is defined by

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where  is the Gamma function.

**Definition 2.** (Riemann-Liouville Fractional Derivative [[50](#_ENREF_50)])**.** The Riemann-Liouville fractional derivative of order  of any function  is defined as follows

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where  is an integer number such that .

**Definition 3.** (Grünwald-Letnikov Fractional Derivative [[50](#_ENREF_50)])**.** The Grünwald-Letnikov fractional derivative of order  of any function  is defined as follows

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This definition of fractional derivative leads to Riemann–Liouville definition when we perform limit operation.

**Definition 4.** (Caputo Fractional Derivative [[50](#_ENREF_50)])**.** The Caputo fractional derivative of order  of any function  is defined as follows

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**Definition 5.** A fractional order linear system in state space form is like:

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where , …, are arbitrary real numbers. If , the system is called commensurate.

**Theorem 1** [[51](#_ENREF_51)]**.** Consider a commensurate fractional order linear system

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where , and  is an arbitrary real number between 0 and 2. The autonomous system is asymptotically stable if the following condition is satisfied:

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where  represents the eigenvalues of matrix .

**Definition 5.** By using the Caputo derivative, a fractional order nonlinear system is defined as:

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where  is the order of fractional derivatives of the system.

By using the Lyapunov direct method, the asymptotic stability of nonlinear fractional order systems can be studied. The Lyapunov direct method for the fractional-order systems is defined in the following theorem [[52](#_ENREF_52)].

**Theorem 2** [[52](#_ENREF_52)]**.** Let  be an equilibrium point for the autonomous fractional order system ‎0(9). Let  be a continuously differentiable function as a Lyapunov function candidate, and  be class-K functions such that

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where . Then the system (9) is asymptotically stable.

**Lemma 1.** [[52](#_ENREF_52)] Let  be a continuously differentiable function. Then, for all 

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**Remark 1.** [[52](#_ENREF_52)] In the case when  , Lemma 1 is still valid. That is,

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In [[53](#_ENREF_53)], Barbalat’s Lemma is developed for fractional order nonlinear systems as follows:

**Theorem 3** [[53](#_ENREF_53)]**.** Let  be a uniformly continuous function on . Assume that there exist two positive constants  and such that  for all  with. Then

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# Problem Statement

Consider the following uncertain fractional order chaotic system

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where  demonstrates the fractional order of the equations,  is measurable state vector of the system,  and  are nonlinear functions and belong to , and it is assumed that , and  stands for the uncertain parameter vector of the system.  is an  vector,  is an external disturbance, and it is assumed that  for all  and the value of  is unknown.  is the control input and the system shows chaotic response for . All functions are assumed to be sufficiently smooth and Lipchitz.  function is -periodic with period *T*, and also the system (15) without control and disturbance has an unstable periodic solution with period . The main goal is to design a control law , such that the chaotic behavior of the system (15) changes to a periodic one which is near to an unstable periodic orbit of the system.

# Stability Analysis and Controller Design

In the mentioned Pyragas method, usually we use a linear delayed feedback control to stabilize the UPO of the system. Here again the concept of the Pyragas technique is used, and it is applied to our problem.

Defining a delayed state as , and substituting it for  in Eq. (15), results in:

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where , and. We subtract Eq. (16) from Eq. (15) to obtain following equations:

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By defining , Eq. (17) can be re-written as (error dynamics):

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To stabilize a periodic orbit of the system, we should obtain the control law  such that the following conditions are satisfied:

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**Theorem 4.** Assume  and , where is the period of the unstable periodic orbit of the chaotic system. If the following control and adaptation laws are applied to system defined by (15), the chaotic behavior of the system is substituted by a regular periodic one.

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where  and  are estimates of  and , is an arbitrary positive constant, and  are positive constants and  is a sufficiently large positive constant. In the above equations, *S* is a sliding surface defined by:

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in which ’s are constants and selected such that an asymptotically stable dynamics for the sliding surface,  is provided.

A simplifying condition, which is very common in controlling fractional order systems, is assumed that all of the system state variables as well as the sliding surface are continuously differentiable and can be measured. This assumption is common especially when a fractional order system is aimed to be controlled [[54](#_ENREF_54), [55](#_ENREF_55)].

**Remark 2.** For the existence of sliding mode, it is necessary that the following manifold be asymptotically stable.

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By applying the time derivative of ’s order to Eq. (27), we have:

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Therefore, the sliding mode dynamics can be obtained as follows

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where  are chosen such that the roots of equation  satisfy the asymptotic stability condition in fractional order linear systems. Using Eq. (8), this condition can be expressed by

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where for  denotes the roots of the above-mentioned equation.

**Proof of Theorem 4.** Consider the following Lyapunov function candidate

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Applying the time derivative of ’s order to the Lyapunov function (31) along with using Lemma 1 results in:

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Substituting for  by Eq. (28) gives:

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Substituting Eq. (18) into Eq. (33) yields:

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It was assumed that , therefore  and consequently:

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Using Eqs. (20)-(22) in inequality (35) yields:

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Now, we add the term  to the right side of inequality (36).

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After some mathematical manipulations inequality (37) can be written as:

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Using Eqs. (24) and (25) in inequality (38) yields:

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Since is bounded, one can conclude that  is also bounded. Using Eq. (23) in inequality (39) results in:

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Integrating both sides of Eq. (40), we have

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One can obtain the following equation from (41)

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Since we assume that the sliding surface is continuously differentiable, it is a uniformly continuous function. Therefore, Using Theorem 3, Inequality (42) indicates that the sliding surface becomes zero as time approaches to infinity. Also, due to asymptotic stability of the origin, the error trajectory converges to zero. Thus, the condition of Eq. (19) is satisfied and the control objective is achieved. So the Theorem 4 has been proved completely.

**Remark 3.** As a special case, if or  is a constant, then  can be selected as:

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**Proof.** If , then , so Eq. (39) will be simplified as follows:

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By substituting Eq. (43) into Eq. (44), one can obtain Eq. (40).

**Remark 4.** Due to use of the sign function in equations of  and , chattering in implementation of control law can occur. To avoid this problem, the sign function can be replaced by saturation as follows [[56](#_ENREF_56)]:

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where  is a small positive number.

# Simulation Results

The fractional order chaotic Duffing system with the following equation is used for simulation:

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where  is the external disturbance and  is the control action. For  and , the fractional order Duffing equation is obtained. By setting , , , , and , the fractional order Duffing equation shows chaotic behavior. Moreover, existence of unstable periodic orbit in fractional order Duffing system is proved [[47](#_ENREF_47)]. One of the UPOs of the above-mentioned system is shown in Figure 1.

We consider  and. Furthermore,  and  are defined as:

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Therefore, Eq. (46) can be rewritten as follows

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The disturbance term is assumed to be  which is bounded by . The parameter  is assumed to be unknown that should be updated by adaptation law (25). The initial conditions are set to , , , and parameters  and  are selected as  and.

The periodic solution with period is considered for stabilization, and the simulation results are shown in Figures 2-4. Figure 2 shows time history of the state variables, Figure 3 represents phase plane of the closed loop control system, and Figure 4 demonstrates time history of the control input and sliding surface. The dashed lines in Figure 2 show the main UPO of the system. As it is observed, the chaotic behavior is substituted by to a periodic orbit close to the UPO of the main system. Note that the controller is applied at . Due to the system uncertainties, external disturbances, and unknown parameters, the main UPO may not be stabilized and the controller action has not converged completely to zero, and the system trajectories converge to a close vicinity of the main UPO. The closer we get to the UPO of the system, the lower control signal will be required because when the trajectory of a chaotic system reaches to the UPO of the system exactly, then the trajectory will remain on the UPO with zero control signal.

# Conclusion

This paper has shown a robust adaptive nonlinear delayed feedback control for a class of uncertain fractional order nonlinear systems. Robustness of the closed loop control system against the system uncertainty and external disturbances is guaranteed by using fractional order sliding mode control method. The control input and adaptation mechanism is constructed from a proper sliding surface via Lyapunov method. The most influential advantage of the proposed method is that it just needs period *T* to derive the controller structure. Because of the system uncertainty and external disturbances, the stabilized orbit is not exactly the UPO of the system. However, it has been tried to stabilize an orbit that is very close to the UPO of the system. Finally, the proposed method is implemented to control a fractional order Duffing system and simulation results are included to illustrate the great performance of the proposed method.

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**Captions:**

**Figure 1.** Unstable periodic orbit of fractional order Duffing system.

**Figure 2.** Stabilizing the 2π-periodic orbit of the fractional order Duffing system: a) first state variable, b) second state variable.

**Figure 3.** Phase plane of the close loop control system.

**Figure 4.** (a) Time history of the control action, (b) Time history of the sliding surface.

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| D:\M.Sc. Thesis\My Papers\Journal Papers\2-after defence\1-control of fractional order chaotic systems by delayed feedback control\re\New folder (3)\chosen_UPO.jpg  **Figure 1.** Unstable periodic orbit of fractional order Duffing system |

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| C:\Users\Bahram\Desktop\papers for submission\6-Communications in nonlinear science\simulations\states_5.jpg  **Figure 2.** Stabilizing the 2π-periodic orbit of the fractional order Duffing system: a) first state variable, b) second state variable  C:\Users\Bahram\Desktop\papers for submission\6-Communications in nonlinear science\simulations\phase plane.jpg  **Figure 3.** Phase plane of the close loop control system |
| C:\Users\Bahram\Desktop\papers for submission\6-Communications in nonlinear science\simulations\control input_2.jpg  **Figure 4.** (a) Time history of the control action, (b) Time history of the sliding surface |